Lecture 31

7.1 Diagonalization

Previous HW problem 3.4 #57

$P = \text{span}\{\vec{v}_1, \vec{v}_n\}$ = plane in $\mathbb{R}^3$

$\vec{v}_3 \neq 0$ vector $\in P^\perp$

$T: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(\vec{x})$ = refl of $\vec{x}$ through P.

What is matrix for T?

Soln: Let $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ = basis for $\mathbb{R}^3$

$\hookrightarrow$ observe $[T]_\beta = \begin{bmatrix} [T(\vec{v}_1)]_\beta & [T(\vec{v}_2)]_\beta & [T(\vec{v}_3)]_\beta \end{bmatrix}$

$$= \begin{bmatrix} [\vec{v}_1]_\beta & [\vec{v}_2]_\beta & [-\vec{v}_3]_\beta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \text{"diagonal matrix"}$$

Def: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is diagonalizable if $\exists$ β for $\mathbb{R}^n$ such that $[T]_\beta$ is a diagonal matrix.

Back to e.g. We have $[T]\vec{x} = S[T]_\beta S^{-1} = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$

$\boxed{\text{Ex}}$ Consider the sequence of integers of $x_0, x_1, x_2, x_3, \ldots$ given by

$x_0 = 0$, $x_1 = 1$, and $x_{n+1} = 3x_n - 2x_{n-1}$

Question: Give a formula for x_n.

Solution: let $\vec{x}_n = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$. Then $\vec{x}_1 = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Also $\vec{x}_{n+1} = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} 3x_n - 2x_{n-1} \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}}_{A\vec{x}_n}$

- Hence $\underset{\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}}{\underset{\shortparallel}{\vec{x}_{n+1}}} = A\vec{x}_n = A(A\vec{x}_{n-1}) = \cdots = A^n \vec{x}_1$

- So now how can we find A^n?

Hint : Consider $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

↳ then $A\vec{v}_1 = \vec{v}_1$, $A\vec{v}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\vec{v}_2$

Side point: $A^n \vec{v}_2 = 2^n \vec{v}_2$

$\vec{x}_0 = \vec{v}_2 - \vec{v}_1$

$\rightarrow A^n \vec{x}_0 = A^n \vec{v}_2 - A^n \vec{v}_1 = 2^n \vec{v}_2 - v_1 = \begin{bmatrix} 2^{n+1} & -1 \\ 2^n & -1 \end{bmatrix}$

$\rightarrow x_n = 2^n - 1$

Alt : Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(\vec{x}) = A\vec{x}$ and $\beta = \{\underset{\vec{v}_1}{\begin{bmatrix}1\\1\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}2\\1\end{bmatrix}}\}$ = basis for $\mathbb{R}^2$

• Then OTOH $A = [T] = S[T]_\beta S^{-1}$

$$= \begin{bmatrix}1&2\\1&1\end{bmatrix}\begin{bmatrix}[T(\begin{smallmatrix}1\\1\end{smallmatrix})]_\beta & [T(\begin{smallmatrix}2\\1\end{smallmatrix})]_\beta\end{bmatrix}\begin{bmatrix}1&2\\1&1\end{bmatrix}^{-1}$$

$$= \begin{bmatrix}1&2\\1&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&2\\1&1\end{bmatrix}$$

• Hence A is similar to $\begin{bmatrix}1&0\\0&2\end{bmatrix}$ and $A^n = \left[S\begin{bmatrix}1&0\\0&2\end{bmatrix}S^{-1}\right]^n$

$$\to A^n = \left[S\begin{bmatrix}1&0\\0&2\end{bmatrix}\cancel{S^{-1}}\right]\left[\cancel{S}\begin{bmatrix}1&0\\0&2\end{bmatrix}\cancel{S^{-1}}\right]\cdots\left[\cancel{S}\begin{bmatrix}1&0\\0&2\end{bmatrix}S^{-1}\right]$$

$$\to A^n = S\begin{bmatrix}1&0\\0&2\end{bmatrix}^n S^{-1} = S\begin{bmatrix}1^n&0^n\\0^n&2^n\end{bmatrix}S^{-1}$$

So $$\vec{x}_{n+1} = \begin{bmatrix}x_{n+1}\\x_n\end{bmatrix} = A\vec{x}_n = A^n\begin{bmatrix}1\\0\end{bmatrix} = \underbrace{\left(\begin{bmatrix}1&2\\1&1\end{bmatrix}\begin{bmatrix}1^n&0^n\\0^n&2^n\end{bmatrix}\begin{bmatrix}1&2\\1&1\end{bmatrix}\right)}_{A^n}\begin{bmatrix}1\\0\end{bmatrix}$$

Equivalent Def : A matrix $A \in M_{n\times n}$ is diagonalizable if $A = P\Lambda P^{-1}$ for some diagonal matrix Λ (i.e. if A is similar to a diagonal matrix)

Def : Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation given by $T(\vec{x}) = A(\vec{x})$. We say $\vec{v} \in \mathbb{R}^n$ is an eigenvector of T with eigenvalue $\lambda \in \mathbb{R}^n$ if $T(\vec{v}) = \lambda\vec{v}$.

(Equivalently $A\vec{v} = \lambda\vec{v}$)

Prop: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be linear. Then the following are equivalent:

(i) T is diagonalizable

(ii) $[T]$ is similar to a diagonal matrix

(iii) $\exists$ basis β for $\mathbb{R}^n$ consisting of e-vecs for T.